

Let $f(x) = e^{-x^2} \cos 2x$.

SCORE: ____ / 25 PTS

[a] Find the first 3 **non-zero** terms of the Maclaurin series for $f(x)$.

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \dots = \underbrace{1 - x^2 + \frac{1}{2}x^4}_{(5)} - \dots$$

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots = \underbrace{1 - 2x^2 + \frac{2}{3}x^4}_{(5)} - \dots$$

$$e^{-x^2} \cos 2x = 1 + (-2-1)x^2 + \left(\frac{2}{3} + 2 + \frac{1}{2}\right)x^4 + \dots$$

$$= \underbrace{1 - 3x^2 + \frac{19}{6}x^4}_{(8)} + \dots$$

[b] Find $f^{(4)}(0)$.

$$\underbrace{(3) \frac{f^{(4)}(0)}{4!} x^4 = \frac{19}{6} x^4}_{(2)}$$

$$f^{(4)}(0) = \underbrace{\frac{19 \cdot 4!}{6}}_{(2)} = 19 \cdot 4 = \underbrace{76}_{(2)}$$

Consider the following statements.

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- (i) If $\sum_{n=1}^{\infty} b_n$ converges, and $a_n < b_n$ for all integers $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges MISSING $0 < a_n$
- (ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges CONTRAPOSITIVE OF ABSOLUTE CONVERGENCE TEST

Which of the statements above are true ? Circle the correct answer below.

[a] neither is true

[b] only (i) is true

[c] only (ii) is true

[d] both are true

Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(3n-2)!x^n}{(n!)^3}$.

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$$\underbrace{\lim_{n \rightarrow \infty} \left| \frac{(3n-2)!}{(n!)^3} \cdot \frac{((n+1)!)^3}{(3(n+1)-2)!} \right|}_{\textcircled{6}} = \lim_{n \rightarrow \infty} \frac{(3n-2)!}{(3n+1)!} \left(\frac{(n+1)!}{n!} \right)^3$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^3}{\underbrace{(3n+1)(3n)(3n-1)}} = \frac{1}{27} = R \quad \textcircled{3}$$

OR

$$\lim_{n \rightarrow \infty} \left| \frac{(3(n+1)-2)! x^{n+1}}{((n+1)!)^3} \cdot \frac{(n!)^3}{(3n-2)! x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(3n+1)!}{(3n-2)!} \left(\frac{n!}{(n+1)!} \right)^3 x \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(3n+1)(3n)(3n-1)}{(n+1)^3} \right| |x| < 1 \rightarrow$$

$$27|x| < 1 \rightarrow |x| < \frac{1}{27} = R$$

Find the first 4 non-zero terms of the Taylor series for $f(x) = \sec x$ about $x = -\frac{\pi}{3}$.

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n	$f^{(n)}(x)$	
0	$\sec x$	ALL ITEMS ② POINTS UNLESS OTHERWISE NOTED
1	$\sec x \tan x$	

$$f^{(n)}\left(-\frac{\pi}{3}\right)$$

$$2$$

$$2(-\sqrt{3}) = -2\sqrt{3}$$

$$2(-\sqrt{3})^2 + 2^2 = 14$$

$$2 \quad \sec x \tan^2 x + \sec^3 x$$

$$3 \quad \sec x \tan^3 x + 2\sec^3 x \tan x + 3\sec^3 x \tan x$$

$$= \sec x \tan^3 x + 5\sec^3 x \tan x$$

$$2(-\sqrt{3})^3 + 5(2)^3(-\sqrt{3})$$

$$= -6\sqrt{3} - 40\sqrt{3}$$

$$= -46\sqrt{3}$$

$$2 - 2\sqrt{3}\left(x - \frac{\pi}{3}\right) + \frac{14}{2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{46\sqrt{3}}{3!}\left(x - \frac{\pi}{3}\right)^3 + \dots$$

$$= 2 - 2\sqrt{3}\left(x + \frac{\pi}{3}\right) + 7\left(x + \frac{\pi}{3}\right)^2 - \frac{23\sqrt{3}}{3}\left(x + \frac{\pi}{3}\right)^3 + \dots$$

①

Determine if the series $\sum_{n=2}^{\infty} \frac{(\cos n)\sqrt{n+1}}{(n-1)^2}$ converges.

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① $0 \leq \left| \frac{(\cos n)\sqrt{n+1}}{(n-1)^2} \right| < \frac{\sqrt{n+1}}{(n-1)^2} = a_n$

LET $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{(n-1)^2} \cdot \frac{n^2}{\sqrt{n}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n+1}{n}} \cdot \left(\frac{n}{n-1}\right)^2 = \sqrt{1} \cdot 1^2 = 1 \neq 0$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ CONVERGES ($p = \frac{3}{2} > 1$) BY p-SERIES TEST. ④

$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{(n-1)^2}$ CONVERGES BY LIMIT COMPARISON TEST

$\sum_{n=1}^{\infty} \left| \frac{(\cos n)\sqrt{n+1}}{(n-1)^2} \right|$ CONVERGES BY COMPARISON TEST

$\sum_{n=1}^{\infty} \frac{(\cos n)\sqrt{n+1}}{(n-1)^2}$ CONVERGES BY ABSOLUTE CONVERGENCE TEST

ALL ITEMS

② POINTS

UNLESS OTHERWISE NOTED

Find the sum of the series $\frac{1}{e} - \frac{1}{2e^2} + \frac{1}{3e^3} - \frac{1}{4e^4} + \frac{1}{5e^5} - \dots$.

SCORE: ____ / 10 PTS

$$= \frac{1}{e} - \frac{\left(\frac{1}{e}\right)^2}{2} + \frac{\left(\frac{1}{e}\right)^3}{3} - \frac{\left(\frac{1}{e}\right)^4}{4} + \frac{\left(\frac{1}{e}\right)^5}{5} - \dots$$

$$= \ln\left(1 + \frac{1}{e}\right)$$

(3) (3)

$$= \ln\left(\frac{e+1}{e}\right)$$

$$= \ln(e+1) - \ln e$$

$$= \ln(e+1) - 1$$

(1)

AJ and CJ are trying to determine if $\sum_{n=1}^{\infty} a_n$ converges, where $a_n = f(n)$.

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They correctly check that f is positive and continuous at all integers $n \geq 1$, and that f' is negative at all integers $n \geq 1$.

They correctly determine that $\int_1^{\infty} f(x) dx$ diverges. AJ concludes that $\sum_{n=1}^{\infty} a_n$ diverges, but CJ thinks they cannot make that conclusion.

Who is right, and why?

CJ - f NEEDS TO BE POSITIVE, CONTINUOUS + DECREASING
FOR ALL REAL NUMBERS IN $[1, \infty)$
(NOT JUST INTEGERS) TO USE INTEGRAL TEST

Find the interval of convergence of the power series $\sum_{n=2}^{\infty} \frac{(2x-6)^n \cos n\pi}{n \ln n}$. $\cos n\pi = (-1)^n$ SCORE: ____ / 35 PTS

$$\lim_{n \rightarrow \infty} \left| \frac{(2x-6)^{n+1} (-1)^{n+1}}{(n+1) \ln(n+1)} \cdot \frac{n \ln n}{(2x-6)^n (-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \frac{\ln n}{\ln(n+1)} \right| |2x-6|$$

$$= 1 \cdot 1 \cdot |2x-6| = |2x-6| < 1$$

NOTE: $\lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$

IF $-1 < 2x-6 < 1$
 $\frac{5}{2} < x < \frac{7}{2}$

IF $x = \frac{5}{2}$

$$\sum_{n=2}^{\infty} \frac{(-1)^n (-1)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$\frac{1}{x \ln x} > 0$, IS CONTINUOUS AND DECREASING ON $[2, \infty)$

SINCE $\left(\frac{1}{x \ln x}\right)' = -\frac{1}{(x \ln x)^2} (\ln x + 1) < 0$ ON $[2, \infty)$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{N \rightarrow \infty} \left| \ln |\ln x| \right|_2^N = \lim_{N \rightarrow \infty} (\ln |\ln N| - \ln |\ln 2|) = \infty$$

SO $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ DIVERGES

IF $x = \frac{7}{2}$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

CONVERGES BY ALTERNATING SERIES TEST
 SINCE $\frac{1}{n \ln n} \rightarrow 0$ AND IS DECREASING

INTERVAL = $\left[\frac{5}{2}, \frac{7}{2}\right]$

ALL ITEMS
 (2) POINTS
 UNLESS
 OTHERWISE
 NOTED